Math 32B, Lecture 4
Multivariable Calculus

Sample Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books, notes, or calculators. Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and to label your axes.

Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

Name: ______________________

UID: ______________________

Section: ______________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

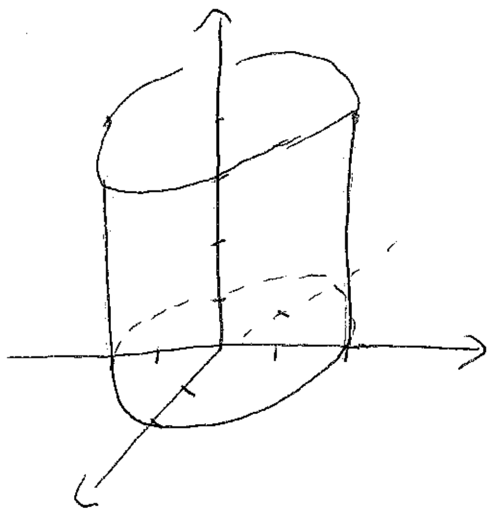
Problem 1.

Consider a cylinder of radius 2 and height 4 with mass density e^{-z}, where z is the height above the base.

(a) [5pts.] What is the mass of the cylinder?

(b) [5pts.] Where does the center of mass of the cylinder lie?

(a)



$$M = \iiint_W \delta(x,y,z)\, dV$$

$$= \int_0^4 \int_0^{2\pi} \int_0^2 e^{-z}\, r\,dr\,d\theta\,dz$$

$$= \int_0^4 e^{-z} dz \cdot \int_0^{2\pi} d\theta \cdot \int_0^2 r\,dr$$

$$= -e^{-z}\Big|_0^4 \cdot 2\pi \cdot \tfrac{1}{2} r^2\Big|_0^2$$

$$= [1-e^{-4}] \cdot 2\pi \cdot 2$$

$$= 4\pi[1-e^{-4}]$$

(b) By symmetry, x and y coordinates are 0.

$$M_{xy} = \iiint_W z e^{-z}\, dV$$

$$= \int_0^4 \int_0^{2\pi} \int_0^2 z e^{-z}\, r\,dr\,d\theta\,dz$$

$$= \int_0^4 z e^{-z} dz \cdot \int_0^{2\pi} d\theta \cdot \int_0^2 r\,dr$$

$$= \left[-ze^{-z} - e^{-z}\right]_0^4 \cdot 2\pi \cdot 2$$

$$= 4\pi(1 - 4e^{-4} - e^{-4})$$

$$= 4\pi(1-5e^{-4})$$

z	e^{-z}
1	$-e^{-z}$
0	e^{-z}

ctd

z-coordinate is $\frac{M_{xy}}{M} = \frac{1-5e^{-4}}{1-e^{-4}}$.

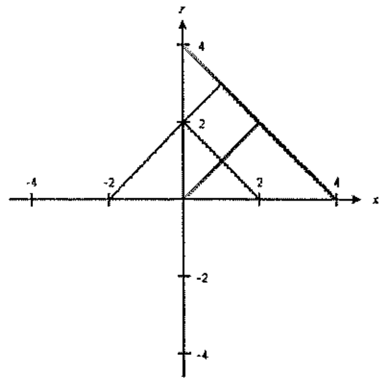
$$\boxed{\left(0, 0, \frac{1-5e^{-4}}{1-e^{-4}}\right)}$$

Problem 2.
Consider the map

$$G(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

which maps into the xy-plane.

(a) [5pts.] Find a domain in the uv-plane which maps to the diamond-shaped region $\mathcal{D}$ cut out by the four lines shown in the xy-plane.



(b) [5pts.] Use your answer from part (a) to evaluate the integral

$$\int\int_{\mathcal{D}} \left((x-y)\sin\left(\frac{\pi}{2}(x+y)\right)\right)^2 dxdy$$

(a) Boundaries

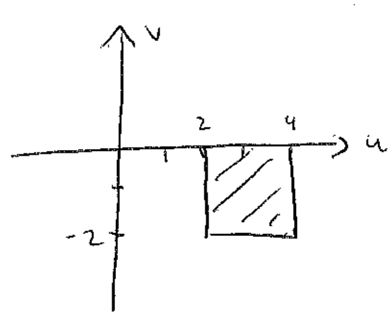
$x+y=2 \rightsquigarrow \frac{u+v}{2} + \frac{u-v}{2} = u = 2$ $\quad \boxed{u=2}$

$x+y=4 \rightsquigarrow \frac{u+v}{2} + \frac{u-v}{2} = u = 4$ $\quad \boxed{u=4}$

$x=y \rightsquigarrow \frac{u+v}{2} = \frac{u-v}{2} \rightsquigarrow \frac{2v}{2} = 0$ $\quad \boxed{v=0}$

$x+2=y \rightsquigarrow \frac{u+v}{2} + 2 = \frac{u-v}{2} \rightsquigarrow \frac{2v}{2} = -2$ $\quad \boxed{v=-2}$

The rectangle $[2,4]\times[-2,0]$ maps to $\mathcal{D}$



$$\text{Jac}(G) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2} , \quad |\text{Jac}(G)| = \frac{1}{2}$$

$$\iint_D \left((x-y) \sin\left(\frac{\pi}{2}(x+y)\right)\right)^2 dx\,dy = \int_2^4 \int_{-2}^0 \left(v \sin\left(\frac{\pi}{2} u\right)\right)^2 \cdot \frac{1}{2}\, dv\, du$$

$$= \int_2^4 \sin^2\left(\frac{\pi}{2} u\right) du \cdot \int_0^2 \frac{1}{2} v^2\, dv$$

$$= \frac{1}{2}\left[u - \frac{1}{2\pi}\sin(\pi u)\right]_2^4 \cdot \frac{1}{6} v^3 \Big|_0^2$$

$$= \frac{1}{2}(2) \cdot \frac{8}{6}$$

$$= \frac{8}{6}$$

$$= \frac{4}{3}$$

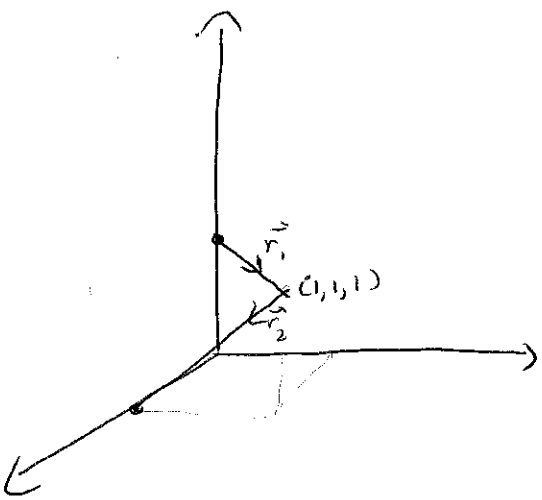
Problem 3.

Calculate the following.

(a) [5pts.] The total electric charge on a piecewise linear wire running from $(0,0,1)$ to $(1,1,1)$ to $(1,0,0)$ with charge density $\delta(x,y,z) = x(y^2 - z)$.

(b) [5pts.] The work done by a vector field $\mathbf{F}(x,y,z) = \langle e^x, e^y, xyz\rangle$ in moving an object from $(0,0,0)$ to $(1,1,\frac{1}{2})$ along thepath $\mathbf{r}(t) = \langle t^2, t, \frac{t}{2}\rangle$.

(a)



$\vec{r}_1(t) = \langle t, t, 1\rangle \qquad 0 \le t \le 1$

$\vec{r}_1{}'(t) = \langle 1, 1, 0\rangle \qquad \|\vec{r}_1{}'(t)\| = \sqrt{2}$

$\vec{r}_2(t) = \langle 1, 1-t, 1-t\rangle$

$\vec{r}_2{}'(t) = \langle 0, -1, -1\rangle \qquad \|\vec{r}_2{}'(t)\| = \sqrt{2}$

$$\int_C \delta(x,y,z)\,ds = \int_0^1 \delta(\vec{r}_1(t)) \cdot \|\vec{r}_1{}'(t)\|\,dt + \int_0^1 \delta(\vec{r}_2(t)) \cdot \|\vec{r}_2{}'(t)\|\,dt$$

$$= \sqrt{2}\int_0^1 t(t^2-1)\,dt + \sqrt{2}\int_0^1 1\left((1-t)^2 - (1-t)\right)dt$$

$$= \sqrt{2}\int_0^1 [t^3 - t]\,dt + \sqrt{2}\int_0^1 [t^2 - t]\,dt$$

$$= \sqrt{2}\left[\frac{1}{4} - \frac{1}{2}\right] + \sqrt{2}\left[\frac{1}{3} - \frac{1}{2}\right]$$

$$= \sqrt{2}\left[\frac{-5}{12}\right]$$

$$= \frac{-5\sqrt{2}}{12}$$

(b) Work $= \int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$

$$= \int_0^1 \langle e^{t^2}, e^t, \tfrac{t^4}{2} \rangle \cdot \langle 2t, 1, \tfrac{1}{2} \rangle \, dt$$

$$= \int_0^1 \left(2te^{t^2} + e^t + \frac{t^4}{4} \right) dt$$

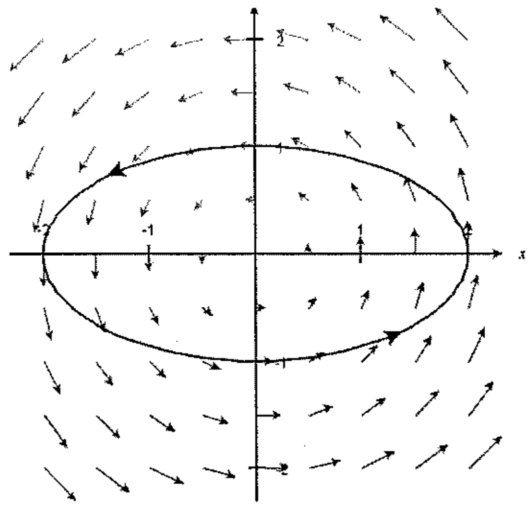
$$= \left(e^{t^2} + e^t + \frac{t^5}{20} \right) \Big|_0^1$$

$$= \left(e + e + \frac{1}{20} \right) - (1 + 1 + 0)$$

$$= 2e - \frac{39}{20}$$

Problem 4.

Consider the vector field $\mathbf{F}$ and the oriented curve $\mathcal{C}$, oriented clockwise, shown below.



(a) [5pts.] Decide whether $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is positive, negative, or zero. Justify your answer.

(b) [5pts.] Decide whether $\mathbf{F}$ is conservative. Justify your answer.

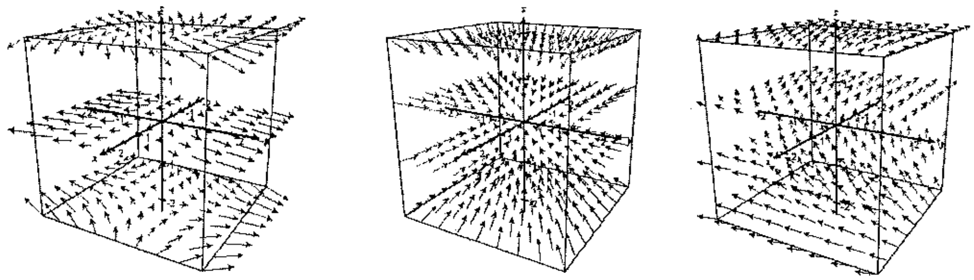
(a) We see that the angles formed by the tangent vectors to the curve and the vectors at the points on the curve in the vector field are acute. So, remembering that $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \int_{t_0}^{t_n} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_{t_0}^{t_n} \|\vec{F}(\vec{r}(t))\| \, \|\vec{r}\,'(t)\| \cos\theta \, dt$ for $\vec{r}(t)$ a parametrization of the curve, and $\cos\theta > 0$ for an acute angle, we expect the line integral to be positive.

(b) $\vec{F}$ is not conservative; if it were, the integral around the closed curve $\mathcal{C}$ would be 0.

Problem 5.

Consider the vector field $\mathbf{F}(x, y, z) = \langle \cos(z), 2y, -x\sin(z)\rangle$

(a) [3pts.] Which of the following is a picture of $\mathbf{F}$? (You do not need to justify your answer, and no partial credit will be given for this problem.)



(b) [3pts.] Find div($\mathbf{F}$) and curl($\mathbf{F}$).

(c) [4pts.] What is the integral of $\mathbf{F}$ over the path $\mathbf{r}(t) = \langle t^7, \sin(2\pi t) - 4, e^{t^2}\rangle$ for $t \in [0, 1]$?

(a) The first picture (most easily seen by looking at the behavior of the z-coordinate.

(b) $\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(\cos z) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(-x\sin z)$

$= 0 + 2 - x\cos z$

$= 2 - x\cos z$

$\operatorname{curl}(\vec{F}) = \left\langle \frac{\partial}{\partial y}(-x\sin z) - \frac{\partial}{\partial z}(2y), \frac{\partial}{\partial z}(\cos z) - \frac{\partial}{\partial x}(-x\sin z), \frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(\cos(z))\right\rangle$

$= \langle 0, -\sin z + \sin z, 0\rangle$

$= \langle 0, 0, 0\rangle$

(c) $\vec{F} = \nabla f$ for $f(x, y, z) = y^2 + x\cos(z)$. So given the path above from $\vec{r}(0) = \langle 0, -4, 1\rangle$ to $\vec{r}(1) = \langle 1, -4, e\rangle$, the line i integral is $\int_{C} \vec{F} \cdot d\vec{r} = F(1, -4, e) - F(0, -4, 1)$

$$= \left(16 + 1(\cos e)\right) - \left(16 + 0\cos(1)\right)$$

$$= \cos(e)$$